

# A SUBORDINATION PRINCIPLE. APPLICATIONS.

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**ABSTRACT.** The subordination principle states roughly: if a property is true for Hardy spaces in some kind of domains in  $\mathbb{C}^n$  then it is also true for the Bergman spaces of the same kind of domains in  $\mathbb{C}^{n-1}$ .

We give applications of this principle to Bergman-Carleson measures, interpolating sequences for Bergman spaces,  $A^p$  Corona theorem and characterization of the zeros set of Bergman-Nevanlinna class.

These applications give precise results for bounded strictly-pseudo convex domains and bounded convex domains of finite type in  $\mathbb{C}^n$ .

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## 1. INTRODUCTION.

Let us start with some definitions. **In all the sequel**, domain will mean bounded domain in  $\mathbb{C}^n$  with smooth  $\mathcal{C}^\infty$  boundary defined by a real valued function  $r \in \mathcal{C}^\infty(\mathbb{C}^n)$ , i.e.  $\Omega = \{z \in \mathbb{C}^n :: r(z) < 0\}$ ,  $\forall z \in \partial\Omega$ ,  $\text{grad}r(z) \neq 0$ ,

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with the defining function  $r$  such that  $\forall z \in \Omega$ ,  $-r(z) \simeq d(z, \Omega^c)$  uniformly on  $\bar{\Omega}$ . (See the beginning of section 2 for the existence of such a function)

Associate to it the "lifted" domain  $\tilde{\Omega}$  in  $(z, w) \in \mathbb{C}^{n+k}$  with defining function

$$\tilde{r}(z, w) := r(z) + |w|^2.$$

This operation keeps the nature of the domain :

- if  $\Omega$  is pseudo-convex,  $\tilde{\Omega}$  is still pseudo-convex ;
- if  $\Omega$  is strictly pseudo-convex, so is  $\tilde{\Omega}$  ;
- if  $\Omega$  is convex, so is  $\tilde{\Omega}$  ;
- if  $\Omega$  is of finite type  $m$ , so is  $\tilde{\Omega}$ .

Moreover we still have  $\forall (z, w) \in \tilde{\Omega}$ ,  $-(r(z) + |w|^2) \simeq d((z, w), \tilde{\Omega}^c)$ .

Let  $dm(z)$  be the Lebesgue measure in  $\mathbb{C}^n$  and  $d\sigma(z)$  be the Lebesgue measure on  $\partial\Omega$ .

For  $z \in \Omega$ , let  $\delta(z) := d(z, \Omega^c) \simeq -r(z)$  be the distance from  $z$  to the boundary of  $\Omega$ .

For  $k \in \mathbb{N}$ , let  $v_k$  be the volume of the unit ball in  $\mathbb{C}^k$  and set

$$\forall z \in \Omega, \quad dm_0(z) := dm(z),$$

$$\forall k \geq 1, \quad \forall z \in \Omega, \quad dm_k(z) := (k+1)v_{k+1}(-r(z))^k dm(z)$$

a weighted Lebesgue measure in  $\Omega$  suitable for our needs. Clearly we have that  $dm_k(z) \simeq \delta(z)^k dm(z)$ .

Let  $\mathcal{U}$  be a neighbourhood of  $\partial\Omega$  in  $\Omega$  such that the normal projection  $\pi$  onto  $\partial\Omega$  is a smooth well defined application.

Define the Bergman, Hardy and Nevanlinna spaces as usual :

**Definition 1.1.** Let  $f$  be a holomorphic function in  $\Omega$  ; we say that  $f \in A_k^p(\Omega)$  if

$$\|f\|_{k,p}^p := \int_{\Omega} |f(z)|^p dm_k(z) < \infty.$$

We say that  $f \in \mathcal{N}_k(\Omega)$  if

$$\|f\|_{\mathcal{N}_k} = \int_{\Omega} \log^+ |f(z)| dm_k(z) < \infty,$$

We say that  $f \in H^p(\Omega)$  if

$$\|f\|_p^p := \sup_{\epsilon > 0} \int_{\{r(z) = -\epsilon\}} |f(\pi(z))|^p d\sigma(z) < \infty.$$

Finally we say that  $f \in \mathcal{N}(\Omega)$  if

$$\|f\|_{\mathcal{N}} = \sup_{\epsilon > 0} \int_{\{r(z) = -\epsilon\}} \log^+ |f(\pi(z))| d\sigma(z) < \infty.$$

This is meaningful because, for  $\epsilon$  small enough, the set  $\{r(z) = -\epsilon\}$  is a smooth manifold in  $\Omega$  contained in  $\mathcal{U}$ .

Now we can state our subordination lemma:

**Lemma 1.2.** (Subordination lemma) Let  $\Omega$  be a domain in  $\mathbb{C}^n$ ,  $\tilde{\Omega}$  its lift in  $\mathbb{C}^{n+k}$  and  $F(z, w) \in H^p(\tilde{\Omega})$ , we have  $f(z) := F(z, 0) \in A_{k-1}^p(\Omega)$  and  $\|f\|_{A_{k-1}^p(\Omega)} \lesssim \|F\|_{H^p(\tilde{\Omega})}$ ;

if  $F(z, w) \in \mathcal{N}(\tilde{\Omega})$ , then  $f(z) := F(z, 0) \in \mathcal{N}_{k-1}(\Omega)$  and  $\|f\|_{\mathcal{N}_{k-1}(\Omega)} \lesssim \|F\|_{\mathcal{N}(\tilde{\Omega})}$ .

A function  $f$ , holomorphic in  $\Omega$ , is in the Bergman space  $A_{k-1}^p(\Omega)$  (resp. in the Nevanlinna Bergman space  $\mathcal{N}_{k-1}(\Omega)$ ) if and only if the function  $F(z, w) := f(z)$  is in the Hardy space  $H^p(\tilde{\Omega})$  (resp. in the Nevanlinna class  $\mathcal{N}(\tilde{\Omega})$ ) and we have  $\|f\|_{A_{k-1}^p} \simeq \|F\|_{H^p(\tilde{\Omega})}$  (resp.  $\|f\|_{\mathcal{N}_{k-1}(\Omega)} \simeq \|F\|_{\mathcal{N}(\tilde{\Omega})}$ ).

In the section 2 we prove the subordination lemma as a consequence of a disintegration of Lebesgue measure.

In the section 3 we introduce the notion of a "good" family of polydiscs  $\mathcal{P}$ , directly inspired by the work of Catlin [14] and introduced in [7] together with a homogeneous hypothesis,  $(Hg)$ . This notion allows us to define geometric Carleson measure, denoted as  $\Lambda(\Omega)$ , for Hardy spaces and denoted as  $\Lambda_k(\Omega)$ , for Bergman spaces and to put it in relation with the Carleson embedding theorem still for these two classes of spaces.

In subsection 3.1 we apply the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding one.

The bounded strictly pseudo-convex domains have Hardy-Carleson embedding property by a result of Hörmander [20], hence they have the Bergman-Carleson embedding property by this result. A direct application of it is the following

**Corollary 1.3.** *A positive Borel measure  $\mu$  in a strictly pseudo-convex domain  $\Omega$  in  $\mathbb{C}^n$  verifies*

$$\forall p \geq 1, \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \lesssim \|f\|_{A_{k-1}^p(\Omega)}^p$$

*iff :*

$$\forall a \in \Omega, \mu(P_a(2)) \lesssim \delta(a)^{n+k},$$

*where  $P_a(2)$  is the polydisc of the good family  $\mathcal{P}$  centered at  $a$  and of "radius" 2.*

This characterization was already proved by Cima and Mercer [15] even for the spaces  $A_{\alpha}^p(\Omega)$  with  $\alpha \geq 0$ . So, in the case where  $\alpha$  is an integer we recover their characterization.

M. Abate and A. Saraco [1] studied Carleson measures in strongly pseudo-convex domains but with a different point of view : instead of using the family of polydiscs to characterize them they use invariant balls.

We have also a characterization for convex domains of finite type, as shown in subsection 2.

**Theorem 1.4.** *Let  $\Omega$  be a convex domain of finite type in  $\mathbb{C}^n$  ; the measure  $\mu$  verifies*

$$(*) \exists p > 1, \exists C_p > 0, \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C_p^p \|f\|_{A_{k-1}^p(\Omega)}^p$$

*iff :*

$$(**) \exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)).$$

*Hence if  $\mu$  verifies  $(*)$  for a  $p > 1$ , it verifies  $(*)$  for all  $q > 1$ .*

Now let  $\Omega$  be a domain in  $\mathbb{C}^n$ . We say that the  $H^p$ -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_m \in H^{\infty}(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then

$$\forall f \in H^p(\Omega), \exists (f_1, \dots, f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the same vein, we say that the  $A_{k-1}^p(\Omega)$ -Corona theorem is true for  $\Omega$  if we have :

$$(1.1) \quad \forall g_1, \dots, g_m \in H^{\infty}(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then

$$\forall f \in A_{k-1}^p(\Omega), \exists (f_1, \dots, f_m) \in (A_{k-1}^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the subsection 5, we apply again the subordination principle, because the  $H^p$  Corona theorem is true in these cases, to get:

**Corollary 1.5.** *We have the  $A_k^p(\Omega)$ -Corona theorem in the following cases :*

- *with  $p = 2$  if  $\Omega$  is a bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;*

- with  $1 < p < \infty$  if  $\Omega$  is a bounded strictly pseudo-convex domain in  $\mathbb{C}^n$ .

In section 4 we define and study the interpolating sequences in a domain  $\Omega$ . We also define the notion of dual bounded sequences in  $H^p(\Omega)$  and in  $A_k^p(\Omega)$ , and applying the subordination principle to the result we proved for  $H^p(\Omega)$  interpolating sequences [7], we get the following theorem.

**Theorem 1.6.** *If  $\Omega$  is a strictly pseudo-convex domain, or a convex domain of finite type in  $\mathbb{C}^n$  and if  $S \subset \Omega$  is a dual bounded sequence of points in  $A_k^p(\Omega)$  then, for any  $q < p$ ,  $S$  is  $A_k^p(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

In the unit ball of  $\mathbb{C}^n$ , we have a better result :

**Theorem 1.7.** *If  $\mathbb{B}$  is the unit ball in  $\mathbb{C}^n$  and if  $S \subset \mathbb{B}$  is a dual bounded sequence of points in  $A_k^p(\mathbb{B})$  then, for any  $q < p$ ,  $S$  is  $A_k^p(\Omega)$  interpolating with the linear extension property.*

Finally in the section 6 we study zeros set for Nevanlinna Bergman functions.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $u$  a holomorphic function in  $\Omega$ . Set  $X := \{z \in \Omega :: u(z) = 0\}$  the zero set of  $u$  and  $\Theta_X := \partial\bar{\partial} \log |u|$  its associated  $(1, 1)$  current of integration.

**Definition 1.8.** *A zero set  $X$  of a holomorphic function  $u$  in the domain  $\Omega$  is in the Blaschke class,  $X \in \mathcal{B}(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \quad \left| \int_{\Omega} (-r(z)) \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty,$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}(\Omega)$  then it is well known [24] that  $X$  is in the Blaschke class of  $\Omega$ .

We do the analogue for the Bergman spaces :

**Definition 1.9.** *A zero set  $X$  of a holomorphic function  $u$  in the domain  $\Omega$  is in the Bergman-Blaschke class,  $X \in \mathcal{B}_k(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \quad \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty,$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}_k(\Omega)$  then  $X$  is in the Bergman-Blaschke class of  $\Omega$  as can be seen again by use of the subordination lemma.

Hence exactly as for the Corona theorem we can set the definitions :

we say that the *Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

And the same for the Bergman spaces :

we say that the *Bergman-Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

We get, by use of the subordination lemma applied to the corresponding Nevanlinna Hardy results,

**Corollary 1.10.** *The Bergman-Blaschke characterization is true in the following cases :*

- *if  $\Omega$  is a strictly pseudo-convex domain in  $\mathbb{C}^n$  ;*
- *if  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$ .*

We stated and proved the subordination lemma for the ball in  $\mathbb{C}^n$  in 1978 [3], and, since then, we gave seminars and conferences about it in the general situation.

As we seen some applications to strictly pseudo-convex domains done here are already known. The applications to the convex domains of finite type are new.

I am grateful to Marco Abate for an interesting discussion on Bergman-Carleson measures in january 2010.

## 2. THE SUBORDINATION LEMMA.

We start with a lemma.

**Lemma 2.1.** *Let  $\Omega$  be a domain in  $\mathbb{C}^n$ , we can always choose the defining function  $s$  such that  $\forall z \in \Omega$ ,  $-s(z) \simeq d(z, \Omega^c)$  uniformly on  $\bar{\Omega}$ .*

*Proof.*

Because  $\text{grad } r(z) \neq 0$  on  $\partial\Omega$ , we have, for a  $\eta > 0$  that  $\forall z :: 0 \leq -r(z) \leq \eta$ ,  $-r(z) \simeq d(z, \Omega^c)$ . Take a  $\mathcal{C}^\infty(\mathbb{R}^+)$  function  $\chi(t)$  such that

$$\forall t \in [0, \eta], \chi(t) = 1, \forall t \geq 2\eta, \chi(t) = 0$$

and set

$$\forall z \in \Omega, s(z) := \chi(-r(z))r(z) - (1 - \chi(-r(z)))d(z, \Omega^c),$$

$$\forall z \notin \Omega, s(z) := r(z).$$

Then clearly  $s(z)$  does the work. ■

We shall need the following lemma on the "disintegration" of the Lebesgue measure on  $\partial\Omega$  with respect to a defining function  $r$ .

**Lemma 2.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary, defined by a function  $r \in \mathcal{C}^\infty$ , i.e.*

$$\Omega := \{x \in \mathbb{R}^n :: r(x) < 0\}, \forall x \in \partial\Omega, \text{grad } r(x) \neq 0.$$

*Then the Lebesgue measure  $\sigma$  on  $\partial\Omega$  is given by*

$$\forall g \in \mathcal{C}(\partial\Omega), \int_{\partial\Omega} g d\sigma = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\{-\eta \leq r(x) < 0\}} \frac{1}{|\text{grad } r(\alpha)|} \tilde{g}(x) dm(x),$$

*where  $\alpha = \pi(x)$  is the normal projection of  $x$  on  $\partial\Omega$  and  $\tilde{g}(x)$  is any continuous extension of  $g$  near  $\partial\Omega$ , for instance  $\tilde{g}(x) := g(\alpha)$ .*

*Proof.*

Set  $\Omega_\eta := \{x \in \Omega :: \delta(x) \leq \eta\}$ . Let  $g \in \mathcal{C}(\partial\Omega)$  and extend  $g$  continuously in  $\Omega_\eta$ . By its very definition, the measure on  $\partial\Omega$  induced by the Lebesgue measure of  $\mathbb{R}^n$  is

$$\int_{\partial\Omega} g d\sigma = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\Omega_\eta} g(x) dm(x).$$

But at a point  $\alpha \in \partial\Omega$ , we have  $r(x) = r(\alpha) + (x - \alpha) \cdot \text{grad } r(\zeta)$ , for a  $\zeta$  on the segment between  $\alpha$  and  $x$ . This implies, because  $r(\alpha) = 0$ ,

$$r(x) = (x - \alpha) \cdot \text{grad } r(\zeta),$$

hence the thickness  $\lambda = \lambda(\alpha)$  of the strip  $\{-\eta \leq r(x) < 0\}$  at a point  $x$ ,  $r(x) = -\eta$ , on the normal at the point  $\alpha$ , i.e.  $x - \alpha = \lambda \text{grad } r(\alpha)$  is

$$\lambda(\alpha) \frac{|\text{grad } r(\alpha) \cdot \text{grad } r(\zeta)|}{|\text{grad } r(\alpha)|} = \eta \Rightarrow \lambda(\alpha) = \frac{|\text{grad } r(\alpha)|}{|\text{grad } r(\alpha) \cdot \text{grad } r(\zeta)|} \eta.$$

Because we want a uniform thickness, we have to correct it :

$$\lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\Omega_\eta} g(x) dm(x) = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\{-\eta \leq r(x) < 0\}} \frac{1}{|\text{grad } r(\alpha)|} g(x) dm(x),$$

because  $\text{grad } r(\zeta) \rightarrow \text{grad } r(\alpha)$  as  $\eta \rightarrow 0$  and this lemma is proved by the Lebesgue dominated convergence theorem.  $\blacksquare$

Now we can prove our subordination lemma 1.2 stated in the introduction.

We copy from [3], and adapt from the ball to this general case. If  $F(z, w) = f(z) \in H^p(\tilde{\Omega})$  then we have

$$\|F\|_p^p := \sup_{\epsilon > 0} \int_{\{\tilde{r}(z, w) = -\epsilon\}} |F(z, w)|^p d\tilde{\sigma}(z, w) < \infty.$$

By lemma 2.2, the Lebesgue measure on  $\partial\tilde{\Omega}$  is

$$\forall g \in \mathcal{C}(\tilde{\Omega}), \int_{\partial\tilde{\Omega}} g(z, w) d\tilde{\sigma}(z, w) = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_{\{-\eta \leq \tilde{r}(z, w) < 0\}} \frac{1}{|\text{grad } \tilde{r}(\alpha)|} g(z, w) dm(z, w),$$

where  $\tilde{r}(z, w) := r(z) + |w|^2$ . To simplify notations set  $A(z, w) := \frac{1}{|\text{grad } \tilde{r}(\alpha)|}$ , then clearly

$$0 < A_1 \leq A(z, w) \leq A_2 < \infty,$$

because  $A(z, w)$  is smooth and  $\tilde{\Omega}$  is compact.

By Fubini we get

$$\begin{aligned} & \int_{\{-\eta \leq \tilde{r}(z, w) < 0\}} A(z, w) g(z, w) dm(z, w) \\ &= \int_{\{z: r(z) < 0\}} \left( \int_{\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}} A(z, w) g(z, w) dm(w) \right) dm(z). \end{aligned}$$

Suppose that the function  $g$  is (real positive) subharmonic in  $w$  for  $z$  fixed. Then

$$\begin{aligned} & \frac{1}{\eta} \int_{\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}} A(z, w) g(z, w) dm(w) \geq \\ & \geq A_1 \frac{1}{\eta} \int_{\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}} g(z, w) dm(w) \geq \\ & \geq g(z, 0) \frac{1}{\eta} \int_{\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}} dm(w), \end{aligned}$$

because  $\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}$  is a corona centered at 0 and  $g$  is subharmonic in  $w$ . Now we have, where the constant  $v_k$  denotes the volume of the unit ball in  $\mathbb{C}^k$ ,

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k: -\eta - r(z) \leq |w|^2 < -r(z)\}} dm(w) = v_k \frac{(-r(z))^k - (-r(z) - \eta)^k}{\eta} \rightarrow k v_k (-r(z))^{k-1}, \quad \eta \rightarrow 0.$$

Hence we get for  $g$  a (real positive) subharmonic function in  $w$  for  $z$  fixed

$$\int_{\partial\tilde{\Omega}} g(z, w) d\tilde{\sigma}(z, w) \geq A_1 v_k k \int_{\{z: r(z) < 0\}} g(z, 0) (-r(z))^{k-1} dm(z) = A_1 \int_{\Omega} g(z, 0) dm_{k-1}(z).$$

Now apply this for  $\tilde{\Omega}_\epsilon := \{r(z) + |w|^2 = -\epsilon\}$  instead of  $\tilde{\Omega}$ , and with  $g(z, w) := |F(z, w)|^p$ , (resp.  $g(z, w) := \log^+ |F(z, w)|$ ) which is pluri-subharmonic, hence in particular subharmonic in  $w$  for  $z$  fixed, and continuous up to  $\partial\tilde{\Omega}_\epsilon$  because  $\epsilon > 0$ . So

$$\int_{\partial\tilde{\Omega}_\epsilon} |F(z, w)|^p d\tilde{\sigma}(z, w) \geq A_1 \int_{\Omega_\epsilon} |F(z, 0)|^p dm_{k-1}(z).$$

Respectively

$$\int_{\partial\tilde{\Omega}_\epsilon} \log^+ |F(z, w)| d\tilde{\sigma}(z, w) \geq A_1 \int_{\Omega_\epsilon} \log^+ |F(z, 0)|^p dm_{k-1}(z).$$

Hence by Fatou's lemma with  $\epsilon \rightarrow 0$ ,

$$A_1 \|F(\cdot, 0)\|_{A_{k-1}^p(\Omega)} = A_1 \int_{\Omega} |F(z, 0)|^p dm_{k-1}(z) \leq \|F\|_{H^p(\tilde{\Omega})}^p,$$

respectively

$$A_1 \|F(\cdot, 0)\|_{\mathcal{N}_{k-1}(\Omega)} = A_1 \int_{\Omega} \log^+ |F(z, 0)| dm_{k-1}(z) \leq \|F\|_{\mathcal{N}(\tilde{\Omega})};$$

hence

$$\|F(\cdot, 0)\|_{A_{k-1}^p(\Omega)} \leq \frac{1}{A_1} \|F\|_{H^p(\tilde{\Omega})}^p,$$

respectively

$$\|F(\cdot, 0)\|_{\mathcal{N}_{k-1}(\Omega)} \leq \frac{1}{A_1} \|F\|_{\mathcal{N}(\tilde{\Omega})}.$$

Now if we set  $f(z) := F(z, 0)$  we get

$$\|f\|_{A_{k-1}^p(\Omega)} \leq \frac{1}{A_1} \|F\|_p^p$$

respectively

$$\|f\|_{\mathcal{N}_{k-1}(\Omega)} \leq \frac{1}{A_1} \|F\|_{\mathcal{N}(\tilde{\Omega})}.$$

Hence if  $F(z, w) := f(z)$  and  $F \in H^p(\tilde{\Omega})$  we have  $f \in A_{k-1}^p(\Omega)$ , respectively  $F \in \mathcal{N}(\tilde{\Omega}) \Rightarrow f \in \mathcal{N}_{k-1}(\Omega)$ .

So we have the first part of the lemma.

Conversely if  $f \in A_{k-1}^p(\Omega)$  (resp.  $f \in \mathcal{N}_{k-1}(\Omega)$ ), setting  $F(z, w) := f(z)$  and reversing the previous computations, using equalities this time,

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} |f(z)|^p dm(w) \rightarrow |f(z)|^p v_k k(-r(z))^{k-1},$$

respectively

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} \log^+ |f(z)| dm(w) \rightarrow \log^+ |f(z)| v_k k(-r(z))^{k-1},$$

we get  $\|F\|_{H^p(\tilde{\Omega})} \leq A_2 \|f\|_{A_{k-1}^p(\Omega)}$  (resp.  $\|F\|_{\mathcal{N}(\tilde{\Omega})} \leq A_2 \|f\|_{\mathcal{N}_{k-1}(\Omega)}$ ). ■

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ , recall the definition of its Szegő projection : this is the orthogonal projection  $P$  from  $L^2(\partial\Omega)$  onto  $H^2(\Omega)$  ; we shall note its kernel by  $S(z, \zeta)$  i.e.

$$\forall f \in L^2(\partial\Omega), Pf(z) = \int_{\partial\Omega} S(z, w) f(\zeta) d\sigma(\zeta).$$

The same way, recall the definition of the Bergman projection : this is the orthogonal projection  $P_k$  from  $L^2(\Omega, dm_k)$  onto  $A_k^2(\Omega)$ , the holomorphic functions on  $\Omega$  still in  $L^2(\Omega, dm_k)$ . We shall note its kernel by  $B_k(z, \zeta)$  i.e.

$$\forall f \in L^2(\Omega, dm_k), P_k f(z) = \int_{\Omega} B_k(z, w) f(\zeta) dm_k(\zeta).$$

Let  $\tilde{\Omega}$  be the lifted domain of  $\Omega$  in  $\mathbb{C}^{n+k}$  ; we shall use the notation

$$\forall z \in \Omega, \tilde{z} := (z, 0) \in \tilde{\Omega}.$$

**Corollary 2.3.** *For any  $a \in \Omega$  the Bergman kernel  $B_{k-1}(z, a)$  and the Szegő kernel  $\tilde{S}((z, w), \tilde{a})$  for the lifted domain  $\tilde{\Omega}$ , verify,*

$$\forall a \in \Omega, \forall z \in \Omega, B_{k-1}(z, a) = \tilde{S}(\tilde{z}, \tilde{a}).$$

Moreover we have

$$\forall a \in \Omega, \|B_{k-1}(\cdot, a)\|_{A_{k-1}^p(\Omega)} \simeq \|\tilde{S}(\cdot, \tilde{a})\|_{H^p(\tilde{\Omega})}.$$

Proof.

Let  $f \in A(\Omega)$  be a holomorphic function in  $\Omega$ , continuous up to  $\partial\Omega$ . Let

$$\forall (z, w) \in \tilde{\Omega}, F(z, w) := f(z).$$

We have

$$\int_{\Omega} f(z) \bar{B}_{k-1}(z, a) dm_{k-1}(z) = f(a) = F(a, 0) = \int_{\partial\tilde{\Omega}} F(z, w) \overline{\tilde{S}((z, w), \tilde{a})} d\sigma(z, w),$$

by the reproducing property of these kernels.

But  $F$  does not depend on  $w$  and  $\overline{\tilde{S}((z, w), \tilde{a})}$  is anti-holomorphic in  $w$  for  $z$  fixed in  $\Omega$ , so

$$\frac{1}{\eta} \int_{\{w \in \mathbb{C}^k : -\eta - r(z) \leq |w|^2 < -r(z)\}} \overline{\tilde{S}((z, w), \tilde{a})} dm(w) \rightarrow \overline{\tilde{S}((z, 0), \tilde{a})} v_k k(-r(z))^{k-1},$$

by the proof of the subordination lemma, hence

$$\int_{\Omega} f(z) \bar{B}_{k-1}(z, a) dm_{k-1}(z) = \int_{\Omega} f(z) \overline{\tilde{S}((z, 0), \tilde{a})} v_k k(-r(z))^{k-1} dm(z) = \int_{\Omega} f(z) \overline{\tilde{S}((z, 0), \tilde{a})} dm_{k-1}(z).$$

So we have

$$\forall f \in A(\Omega), \int_{\Omega} f(z) (\overline{\tilde{S}((z, 0), \tilde{a})} - \bar{B}_{k-1}(z, a)) dm_{k-1}(z) = 0,$$

hence  $\overline{\tilde{S}((z, 0), \tilde{a})} - B_{k-1}(z, a) \perp A(\Omega)$  in  $A_{k-1}^2(\Omega)$ . But  $\tilde{S}((z, 0), \tilde{a}) - B_k(z, a)$  is holomorphic in  $z$ , hence

$$\forall z \in \Omega, \tilde{S}((z, 0), \tilde{a}) = B_{k-1}(z, a).$$

The second part is a direct application of the first part in the subordination lemma 1.2. ■

### 3. GEOMETRIC CARLESON MEASURES AND $p$ -CARLESON MEASURES.

In order to define precisely the geometric Carleson measures, we need the notion of a "good" family of polydiscs, directly inspired by the work of Catlin [14] and introduced in [7].

Let  $\mathcal{U}$  be a neighbourhood of  $\partial\Omega$  in  $\Omega$  such that the normal projection  $\pi$  onto  $\partial\Omega$  is a smooth well defined application.

Let  $\alpha \in \partial\Omega$  and let  $b(\alpha) = (L_1, L_2, \dots, L_n)$  be an orthonormal basis of  $\mathbb{C}^n$  such that  $(L_2, \dots, L_n)$  is a basis of the tangent complex space  $T_{\alpha}^{\mathbb{C}}$  of  $\partial\Omega$  at  $\alpha$ ; hence  $L_1$  is the complex normal at  $\alpha$  to  $\partial\Omega$ . Let  $m(\alpha) = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$  be a multi-index at  $\alpha$  with  $m_1 = 1, \forall j \geq 2, m_j \geq 2$ . For  $a \in \mathcal{U}$  and  $t > 0$  set  $\alpha = \pi(a)$  and  $P_a(t) := \prod_{j=1}^n t D_j$ , the polydisc such that  $t D_j$  is the disc centered at  $a$ , parallel to  $L_j \in b(\alpha)$ , with radius  $t |r(a)|^{1/m_j}$  (recall that we have  $|r(a)| \simeq \delta(a)$ ).

Set  $b(a) := b(\pi(a))$ ,  $m(a) := m(\pi(a))$ , for  $a \in \mathcal{U}$ .

This way we have a family of polydiscs  $\mathcal{P} := \{P_a(t)\}_{a \in \mathcal{U}}$  defined by the family of basis  $\{b(a)\}_{a \in \mathcal{U}}$ , the family of multi-indices  $\{m(a)\}_{a \in \mathcal{U}}$  and the number  $t$ . Notice that the polydisc  $P_a(2)$  always overflows the domain  $\Omega$ .



It will be useful to extend this family to the whole of  $\Omega$ . In order to do so let  $(z_1, \dots, z_n)$  be the canonical coordinates system in  $\mathbb{C}^n$  and for  $a \in \Omega \setminus \mathcal{U}$ , let  $P_a(t)$  be the polydisc of center  $a$ , of sides parallel to the axis and radius  $t\delta(a)$  in the  $z_1$  direction and  $t\delta(a)^{1/2}$  in the other directions. So the points  $a \in \Omega \setminus \mathcal{U}$  have automatically a "minimal" multi-index  $m(a) = (1, 2, \dots, 2)$ .

Now we can set

**Definition 3.1.** We say that  $\mathcal{P}$  is a "good family" of polydiscs for  $\Omega$  if the  $m_j(a)$  are uniformly bounded on  $\Omega$  and if it exists  $\delta_0 > 0$  such that all the polydiscs  $P_a(\delta_0)$  of  $\mathcal{P}$  are contained in  $\Omega$ . In this case we call  $m(a)$  the multi-type at  $a$  of the family  $\mathcal{P}$ .

We notice that, for a good family  $\mathcal{P}$ , by definition the multi-type is always finite. Moreover there is no regularity assumptions on the way that the basis  $b(a)$  varies with respect to  $a \in \Omega$ .

We can see easily that there are always good families of polydiscs in a domain  $\Omega$  in  $\mathbb{C}^n$  : for a point  $a \in \Omega$ , take any orthonormal basis  $b(a) = (L_1, L_2, \dots, L_n)$ , with  $L_1$  the complex normal direction, and the "minimal" multitype  $m(a) = (1, 2, \dots, 2)$ . Then, because the level sets  $\partial\Omega_a$  are uniformly of class  $\mathcal{C}^2$  and compact, we have the existence of a uniform  $\delta_0 > 0$  such that the family  $\mathcal{P}$  is a good one. As seen in [7], in the strictly pseudo-convex domains, this family with "minimal" multi-type is the right one.

We can give the definitions relative to Carleson measures.

**Definition 3.2.** A positive borelian measure  $\mu$  on  $\Omega$  is a geometric Carleson measure,  $\mu \in \Lambda(\Omega)$ , if  $\exists C = C_\mu > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C\sigma(\partial\Omega \cap P_a(2))$ .

**Definition 3.3.** A positive borelian measure  $\mu$  on  $\Omega$  is a  $p$ -Carleson measure in  $\Omega$  if

$$\exists C > 0 :: \forall f \in H^p(\Omega), \int_{\Omega} |f(z)|^p d\mu(z) \leq C^p \|f\|_{H^p(\Omega)}^p.$$

And analogously for the Bergman spaces.

**Definition 3.4.** A positive borelian measure  $\mu$  on  $\Omega$  is a  $k$ -geometric Bergman-Carleson measure,  $\mu \in \Lambda_k(\Omega)$ , if

$$\exists C = C_\mu > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)).$$

Notice the gap  $k \rightarrow k - 1$ .

**Definition 3.5.** A positive borelian measure  $\mu$  is  $(p, k)$ -Bergman-Carleson measure in  $\Omega$  if

$$\exists C > 0 :: \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f(z)|^p d\mu(z) \leq C^p \|f\|_{A_{k-1}^p(\Omega)}^p.$$

**Definition 3.6.** We shall say that the domain  $\Omega$  has the  $p$ -Carleson embedding property,  $p$ -CEP, if

$$\forall \mu \in \Lambda(\Omega), \exists C = C_\mu > 0 :: \forall f \in H^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{H^p(\Omega)}^p.$$

And the same for the Bergman spaces.

**Definition 3.7.** We shall say that the domain  $\Omega$  has the  $(p, k)$ -Bergman-Carleson embedding property,  $(p, k)$ -BCEP, if

$$\forall \mu \in \Lambda_k(\Omega), \exists C = C_{\mu, p} > 0 :: \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C \|f\|_{A_{k-1}^p(\Omega)}^p.$$

### 3.1. The subordination lemma applied to Carleson measures.

We shall fix  $k \in \mathbb{N}$  and lift the measure on the domain  $\tilde{\Omega} := \{\tilde{r}(z, w) := r(z) + |w|^2 < 0\}$ , with  $w = (w_1, \dots, w_k) \in \mathbb{C}^k$ . We already know how to lift a function, the lifted measure  $\tilde{\mu}$  of a measure  $\mu$  is just

$$\tilde{\mu} := \mu \otimes \delta,$$

with  $\delta$  the delta Dirac measure of the origin in  $\mathbb{C}^k$ . We shall need a lemma linking Bergman and Hardy geometric Carleson measures.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ ,  $\tilde{\Omega}$  be its lift in  $\mathbb{C}^{n+k}$ , and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$ , we have the definition :

**Definition 3.8.** *We shall say that the good family of polydiscs  $\tilde{\mathcal{P}}$  on the domain  $\tilde{\Omega}$  is "homogeneous" if*

$$(Hg) \exists t > 0, \exists C > 0 :: \forall a \in \tilde{\Omega}, \Omega \cap \tilde{P}_a(2) \neq \emptyset, \\ \forall b \in \Omega \cap \tilde{P}_a(2), \tilde{P}_b(t) \supset \tilde{P}_a(2) \text{ and } \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_b(t)) \leq C\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_a(2)),$$

where  $\Omega = \tilde{\Omega} \cap \{w = 0\} \subset \tilde{\Omega}$ .

Naturally the domain  $\Omega$  is equipped with the family  $\mathcal{P}$  induced by  $\tilde{\mathcal{P}}$  the following way

$$\forall a \in \Omega, P_a(u) := \tilde{P}_{(a,0)}(u) \cap \{w = 0\},$$

which is easily seen to be a good family for  $\Omega$ .

As examples we have the strictly pseudo-convex domains and the convex domains of finite type, because both are domains of homogeneous type in the sense of Coifman-Weiss [16].

**Lemma 3.9.** *Let  $(\Omega, \tilde{\Omega})$  be as above and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypothesis (Hg). The measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$  iff the measure  $\tilde{\mu}$  is a geometric Carleson measure in  $\tilde{\Omega}$ .*

*Proof.*

Suppose that  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ , we want to show :

$$\exists C > 0 :: \forall (a, b) \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)) \leq C\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)),$$

with  $\tilde{P}_c$  the polydisc of center  $c = (a, b) \in \tilde{\Omega}$  of the family  $\tilde{\mathcal{P}}$ . Let us see first the case where  $b = 0$ , i.e.  $(a, b) = (a, 0) \in \Omega \subset \tilde{\Omega}$ . Then, by definition of  $\tilde{\mu}$ , we have

$$\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,0)}(2)) = \mu(\Omega \cap P_a(2)).$$

On the other hand, we have, exactly as in the proof of the subordination lemma,

$$\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,0)}(2)) \simeq \int_{\Omega \cap P_a(2)} kv_k(-r(z))^{k-1} dm(z) = m_{k-1}(\Omega \cap P_a(2)).$$

But if  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ , we have

$$\exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq Cm_{k-1}(\Omega \cap P_a(2)),$$

so

$$\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,0)}(2)) = \mu(\Omega \cap P_a(2)) \leq Cm_{k-1}(\Omega \cap P_a(2)) \simeq C\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,0)}(2)).$$

Now take a general  $\tilde{P}_{(a,b)}(2)$ . In order for  $\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2))$  to be non zero, we must have

$$\tilde{P}_{(a,b)}(2) \cap \{w = 0\} \neq \emptyset \Rightarrow \exists (c, 0) \in \tilde{P}_{(a,b)}(2) \cap \{w = 0\}.$$

By the (Hg) hypothesis, this means that we have  $\tilde{P}_{(c,0)}(t) \supset \tilde{P}_{(a,b)}(2)$  with the uniform control  $\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(c,0)}(t)) \lesssim \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2))$ .

We apply the above inequality

$$\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)) \leq \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(c,0)}(t)) \leq Cm_k(\Omega \cap P_c(t)) = C\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(c,0)}(t)) \lesssim \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)),$$

hence  $\tilde{\mu}$  is a geometric Carleson measure on  $\tilde{\Omega}$ .

Conversely suppose that  $\tilde{\mu}$  is a geometric Carleson measure on  $\tilde{\Omega}$ , this means

$$\forall (a, b) \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)) \leq C\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{(a,b)}(2)),$$

hence, in particular for  $b = 0$ ,

$$\forall a \in \Omega, \tilde{\mu}(\tilde{\Omega} \cap P_{(a,0)}(2)) \leq C\tilde{\sigma}(\tilde{\Omega} \cap P_{(a,0)}(2)),$$

but then, by definition of  $\tilde{\mu}$  and with the previous computation of  $\tilde{\sigma}(\tilde{\Omega} \cap P_{(a,0)}(2))$ , we get

$$\forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq Cm_{k-1}(\Omega \cap P_a(2)),$$

hence the measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure in  $\Omega$ .  $\blacksquare$

Now we shall use the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding one.

**Theorem 3.10.** *Let  $(\Omega, \tilde{\Omega})$  be as usual and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypotheses (Hg). If the lifted domain  $\tilde{\Omega}$  has the  $p$ -CEP then  $\Omega$  has the  $(p, k)$ -BCEP.*

*Proof.*

Suppose the positive measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure ; by the previous lemma, we have that the lifted measure  $\tilde{\mu}$  is a geometric Carleson measure in  $\tilde{\Omega}$ . By the  $p$ -CEP we have

$$\forall F \in H^p(\tilde{\Omega}), \int_{\tilde{\Omega}} |F|^p d\tilde{\mu} \leq C_\mu^p \|F\|_{H^p(\tilde{\Omega})}^p.$$

Choose  $f(z) \in A_{k-1}^p(\Omega)$  and set  $\forall (z, w) \in \tilde{\Omega}$ ,  $F(z, w) = f(z)$ . By the subordination lemma we have

$$\|f\|_{A_{k-1}^p(\Omega)} \simeq \|F\|_{H^p(\tilde{\Omega})},$$

and by definition of  $\tilde{\mu}$ , we have

$$\int_{\Omega} |f|^p d\mu = \int_{\tilde{\Omega}} |F|^p d\tilde{\mu} \leq C_\mu^p \|F\|_{H^p(\tilde{\Omega})}^p \lesssim \|f\|_{A_{k-1}^p(\Omega)}^p,$$

hence  $\mu$  is a  $(k, p)$ -Bergman-Carleson measure in  $\Omega$ .  $\blacksquare$

**Theorem 3.11.** *Let  $(\Omega, \tilde{\Omega})$  be as usual and suppose that  $\tilde{\Omega}$  is equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  which verifies the hypotheses (Hg). If  $p$ -Carleson implies geometric Carleson in  $\tilde{\Omega}$ , then  $(p, k)$ -Bergman-Carleson implies geometric  $k$ -Bergman-Carleson in  $\Omega$ .*

*Proof.*

If the positive measure  $\mu$  is  $(p, k)$ -Bergman-Carleson in  $\Omega$  then  $\tilde{\mu}$  is a  $p$ -Carleson measure in  $\tilde{\Omega}$  by lemma 3.9 hence a geometric Carleson measure in  $\tilde{\Omega}$  by the assumption of the theorem. Then applying lemma 3.9 we get that  $\mu$  is a  $k$ -geometric Carleson measure in  $\Omega$  hence the theorem.  $\blacksquare$

**Remark 3.12.** *The definition of geometric Carleson measures depends on the chosen good family of polydiscs on the domain ; the theorem asserts the equivalence of properties between a domain  $\Omega$  and its lift  $\tilde{\Omega}$ . The fact that a lifted domain  $\tilde{\Omega}$  equipped with a good family of polydiscs  $\tilde{\mathcal{P}}$  has the Carleson embedding property has to be proved directly but if it has the  $p$ -CEP then  $\Omega$  equipped with the induced family  $\mathcal{P}$  has the  $(p, k)$ -BCEP without any further proof.*

### 3.2. Application to strictly pseudo-convex domains.

**Corollary 3.13.** *Let  $\Omega$  be a strictly pseudo-convex domain equipped with its minimal good family of polydiscs, then  $\Omega$  has the  $(p, k)$  Bergman Carleson embedding property.*

*Proof.*

The domain  $\Omega$  equipped with its minimal good family has the  $p$ -CEP by Hörmander [20], hence we can apply theorem 3.11.  $\blacksquare$

This corollary gives a characterization of the  $(p, k)$ -Bergman-Carleson measures of the strictly pseudo-convex domains. Let  $\Omega$  be a strictly pseudo-convex domain and  $\tilde{\Omega}$  its lift in  $\mathbb{C}^{n+k}$ . Let  $\tilde{\mathcal{P}}$  be its minimal good family of polydiscs in  $\tilde{\Omega}$  ; one can see easily that the induced family of polydiscs  $\mathcal{P}$  on  $\Omega$  is again the minimal good family of polydiscs. We have this characterization :

**Corollary 3.14.** *A positive Borel measure  $\mu$  in a strictly pseudo-convex domain in  $\mathbb{C}^n$  is a  $(p, k)$ -Bergman-Carleson measure iff :*

$$\forall a \in \Omega, \mu(P_a(2)) \lesssim \delta(a)^{n+k}.$$

*This means that it is a characterization of the measures such that*

$$\forall p \geq 1, \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \lesssim \|f\|_{A_{k-1}^p(\Omega)}^p.$$

*In particular this characterization is independent of  $p \geq 1$ .*

*Proof.*

Let  $\tilde{\Omega}$  be the lift of  $\Omega$  in  $\mathbb{C}^{n+k}$  and  $\tilde{\mu}$  be the lift of  $\mu$  on  $\tilde{\Omega}$ .

Suppose that  $\mu$  is a  $(p, k)$ -Bergman Carleson measure in  $\Omega$ , then  $\tilde{\mu}$  is a  $p$ -Carleson measure in  $\tilde{\Omega}$  by lemma 3.9 then by a theorem of Hörmander [20] the  $p$ -Carleson measures are precisely the geometric ones in  $\tilde{\Omega}$ , hence we have

$$\forall \tilde{a} \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \lesssim \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)).$$

Now let  $a \in \Omega$ ,  $\tilde{a} := (a, 0) \in \tilde{\Omega}$  then a classical computation gives  $\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \lesssim \tilde{\delta}(\tilde{a})^{n+k} = \delta(a)^{n+k}$ . By the definition of  $\tilde{\mu}$  we have

$$\delta(a)^{n+k} \gtrsim \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) = \mu(\tilde{P}_{\tilde{a}}(2) \cap \Omega) = \mu(P_a(2) \cap \Omega),$$

so  $\forall a \in \Omega$ ,  $\mu(P_a(2) \cap \Omega) \lesssim \delta(a)^{n+k}$ .

Now suppose that  $\forall a \in \Omega$ ,  $\mu(P_a(2) \cap \Omega) \lesssim \delta(a)^{n+k}$  then we have, by the definition of  $\tilde{\mu}$ , with  $\tilde{a} := (a, 0) \in \tilde{\Omega}$ ,

$$\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \leq \tilde{\delta}(\tilde{a})^{n+k} \simeq \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)).$$

Doing exactly as in the proof of lemma 3.9 we have the same inequality with a bigger constant for all  $\tilde{a} \in \tilde{\Omega}$ , hence  $\tilde{\mu}$  is a geometric Carleson measure in  $\tilde{\Omega}$ . So by Hörmander [20],  $\tilde{\mu}$  is a  $p$ -Carleson measure in  $\tilde{\Omega}$  hence we have the embedding

$$\forall F \in H^p(\tilde{\Omega}), \int_{\tilde{\Omega}} |F|^p d\tilde{\mu} \lesssim \|F\|_{H^p(\tilde{\Omega})}^p.$$

Now we take  $f \in A_{k-1}^p(\Omega)$  and we set  $\forall (z, w) \in \tilde{\Omega}$ ,  $F(z, w) := f(z)$  by the subordination lemma we have  $\|F\|_{H^p(\tilde{\Omega})} \simeq \|f\|_{A_{k-1}^p(\Omega)}$  and

$$\int_{\tilde{\Omega}} |F|^p d\tilde{\mu} = \int_{\Omega} |f|^p dm_{k-1} \lesssim \|F\|_{H^p(\tilde{\Omega})}^p \simeq \|f\|_{A_{k-1}^p(\Omega)}^p. \quad \blacksquare$$

Cima and Mercer [15] characterized the Carleson measures for the spaces  $A_{\alpha}^p(\Omega)$  for  $\Omega$  strictly pseudo-convex, and with  $\alpha \geq 0$ . In the case where  $\alpha$  is an integer we recover their characterization, because one has easily, when  $\Omega$  is a strictly pseudo-convex domain, that  $P_a(2) \cap \Omega \simeq W(\pi(a), \delta(a))$  where  $W(\zeta, h)$  is the classical Carleson window in  $\Omega$ .

**Remark 3.15.** *In the case of the unit ball  $\Omega$  of  $\mathbb{C}^n$ ,  $\tilde{\Omega} \subset \mathbb{C}^{n+1}$  N. Varopoulos showed me an alternative proof for the fact that  $F(z, w) \in H^p(\tilde{\Omega}) \Rightarrow F(z, 0) \in A_p(\Omega)$  : the Lebesgue measure on  $\{w = 0\} \cap \tilde{\Omega}$  is easily seen to be a geometric Carleson measure in  $\tilde{\Omega}$ , hence by the Carleson-Hörmander embedding theorem [20] we have*

$$\int_{\Omega} |F(z, 0)|^p dm(z) \leq C \|F\|_{H^p(\tilde{\Omega})}^p,$$

*and the assertion. Of course this is still valid in codimension  $k \geq 1$ , with the weighted Lebesgue measure on  $\Omega$ , and for strictly pseudo-convex domains because the Carleson-Hörmander embedding theorem is still valid there. But this is just one direction of the lemma, it works only if there is a Carleson embedding theorem and this proof is much less elementary than the previous one.*

*In fact we can reverse things and say that one part of the subordination lemma asserts that the weighted Lebesgue measure on  $\Omega$  is always a Carleson measure in  $\tilde{\Omega}$ ,  $\Omega$  strictly pseudo-convex or not.*

### 3.3. Application to convex domains of finite type in $\mathbb{C}^n$ .

In [7] we prove a Carleson embedding theorem for the convex domains of finite type in  $\mathbb{C}^n$ .

**Theorem 3.16.** *Let  $\Omega$  be a convex domain of finite type in  $\mathbb{C}^n$  ; if the measure  $\mu$  is a geometric Carleson measure we have*

$$\forall p > 1, \exists C_p > 0, \forall f \in H^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C_p^p \|f\|_{H^p}^p.$$

*Conversely if the positive measure  $\mu$  is  $p$ -Carleson for a  $p \in [1, \infty[$ , then it is a geometric Carleson measure, hence it is  $q$ -Carleson for any  $q \in ]1, \infty[$ .*

We already know that if  $\Omega$  is a convex domain of finite type, so is  $\tilde{\Omega}$  with the same type. Moreover the hypothesis (Hg) is true for these domains equipped with a (slightly modified) McNeal family of polydiscs, so we can apply what precedes in this case to get from the Carleson embedding theorem the Bergman-Carleson embedding one.

**Theorem 3.17.** *Let  $\Omega$  be a convex domain of finite type in  $\mathbb{C}^n$  ; if the measure  $\mu$  is a  $k$ -geometric Bergman-Carleson measure, i.e.*

$$\exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)),$$

*we have*

$$\forall p > 1, \exists C_p > 0, \forall f \in A_{k-1}^p(\Omega), \int_{\Omega} |f|^p d\mu \leq C_p^p \|f\|_{A_{k-1}^p(\Omega)}^p.$$

*Conversely if the positive measure  $\mu$  is  $(p, k)$ -Bergman-Carleson for a  $p \in [1, \infty[$ , then it is a  $k$ -geometric Bergman-Carleson measure, hence it is  $(q, k)$ -Bergman-Carleson for any  $q \in ]1, \infty[$ .*

#### 4. INTERPOLATING SEQUENCES FOR BERGMAN SPACES.

For  $a \in \Omega$ , let  $k_a(z) := S(z, a)$  denotes the Szegö kernel of  $\Omega$  at the point  $a$ . It is also the reproducing kernel for  $H^2(\Omega)$ , i.e.

$$\forall a \in \Omega, \forall f \in H^2(\Omega), f(a) = \int_{\partial\Omega} f(z) \bar{k}_a(z) d\sigma(z) = \langle f, k_a \rangle.$$

Set  $\|k_a\|_p := \|k_a\|_{H^2(\Omega)}$  and:

**Definition 4.1.** *We say that the sequence  $\Lambda$  of points in  $\Omega$  is  $H^p(\Omega)$  interpolating if*

$$\forall \lambda \in \ell^p(\Lambda), \exists f \in H^p(\Omega) :: \forall a \in \Lambda, f(a) = \lambda_a \|k_a\|_{p'},$$

*with  $p'$  the conjugate exponent for  $p$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ .*

We say that  $\Lambda$  has the linear extension property if  $\Lambda$  is  $H^p(\Omega)$  interpolating and if moreover there is a bounded linear operator  $E : \ell^p(\Lambda) \rightarrow H^p(\Omega)$  making the interpolation, i.e.

$$\forall \lambda \in \ell^p(\Lambda), E(\lambda) \in H^p(\Omega), \forall a \in \Lambda, E(\lambda)(a) = \lambda_a \|k_a\|_{p'}.$$

A weaker notion is the dual boundedness:

**Definition 4.2.** *We shall say that the sequence  $\Lambda$  of points in  $\Omega$  is dual bounded in  $H^p(\Omega)$  if there is a bounded sequence of elements in  $H^p(\Omega)$ ,  $\{\rho_a\}_{a \in \Lambda} \subset H^p(\Omega)$  which dualizes the associated sequence of reproducing kernels, i.e.*

$$\exists C > 0 :: \forall a \in \Lambda, \|\rho_a\|_p \leq C, \forall a, c \in \Lambda, \langle \rho_a, k_c \rangle = \delta_{a,c} \|k_c\|_{p'}.$$

If  $\Lambda$  is  $H^p(\Omega)$  interpolating then it is dual bounded in  $H^p(\Omega)$  : just interpolate the elements of the basic sequence in  $\ell^p(\Lambda)$ .

The converse is the crux of the characterization by Carleson [13] of  $H^\infty(\mathbb{D})$  interpolating sequences and the same by Shapiro & Shields [23] for  $H^p(\mathbb{D})$  interpolating sequences in  $\mathbb{D}$ .

We do the same for the Bergman spaces.

For  $k \in \mathbb{N}$  and  $a \in \Omega$ , let  $b_{k,a}(z) := B_k(z, a)$  denotes the Bergman kernel of  $\Omega$  at the point  $a$ . It is also the reproducing kernel for  $A_k^2(\Omega)$ , i.e.

$$\forall a \in \Omega, \forall f \in A_k^2(\Omega), f(a) = \int_{\Omega} f(z) \bar{b}_{k,a}(z) dm_k(z) = \langle f, b_{k,a} \rangle.$$

Now we set  $\|b_{k,a}\|_p := \|b_{k,a}\|_{A_k^p(\Omega)}$  and:

**Definition 4.3.** We say that the sequence  $\Lambda$  of points in  $\Omega$  is  $A_k^p(\Omega)$  interpolating if

$$\forall \lambda \in \ell^p(\Lambda), \exists f \in A_k^p(\Omega) :: \forall a \in \Lambda, f(a) = \lambda_a \|b_{k,a}\|_{p'},$$

with  $p'$  the conjugate exponent for  $p$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ .

We say that  $\Lambda$  has the linear extension property if  $\Lambda$  is  $A_k^p(\Omega)$  interpolating and if moreover there is a bounded linear operator  $E : \ell^p(\Lambda) \rightarrow A_k^p(\Omega)$  making the interpolation.

**Definition 4.4.** We shall say that the sequence  $\Lambda$  of points in  $\Omega$  is dual bounded in  $A_k^p(\Omega)$  if there is a bounded sequence of elements in  $A_k^p(\Omega)$ ,  $\{\rho_a\}_{a \in \Lambda} \subset A_k^p(\Omega)$  which dualizes the associated sequence of reproducing kernels, i.e.

$$\exists C > 0 :: \forall a \in \Lambda, \|\rho_a\|_p \leq C, \forall a, c \in \Lambda, \langle \rho_a, b_{k,c} \rangle = \delta_{a,c} \|b_{k,a}\|_{p'}.$$

Again if  $\Lambda$  is  $A_k^p(\Omega)$  interpolating then it is dual bounded in  $A_k^p(\Omega)$  : just interpolate the elements of the basic sequence in  $\ell^p(\Lambda)$ .

#### 4.1. Case of the unit disc $\mathbb{D}$ in $\mathbb{C}$ .

In that case the interpolating sequences for  $H^\infty(\mathbb{D})$  where characterized by Carleson [13] and for  $H^p(\mathbb{D})$  by Shapiro & Shields [23]. The interpolating sequences for the Bergman spaces  $A_k^p(\mathbb{D})$  were characterized by Seip [22].

In these cases it appears that dual boundedness implies interpolation. For Hardy spaces dual boundedness is easily seen to be equivalent to the Carleson condition and for Bergman spaces, it is proved by Schuster & Seip [21].

#### 4.2. General case.

We shall apply the subordination lemma to interpolating sequences in general domains  $\Omega$ . Let  $\tilde{\Omega}$  be the lifted domain in  $\mathbb{C}^{n+k}$  associated to  $\Omega$ . Let  $\tilde{\Lambda}$  be the sequence  $\Lambda$  viewed in  $\tilde{\Omega}$ ,  $\tilde{\Lambda} := \Lambda \subset \Omega \subset \tilde{\Omega}$ . Let us denote by  $k_{\tilde{a}}(z, w) := S((z, w), \tilde{a})$  the Szegő kernel of  $\tilde{\Omega}$ , for  $\tilde{a} = (a, 0)$ .

**Theorem 4.5.** Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\tilde{\Omega}$  its lift to  $\mathbb{C}^{n+k}$ . If  $\Lambda \subset \Omega$  is a sequence of points in  $\Omega$ , let  $\tilde{\Lambda}$  be the sequence  $\Lambda$  viewed in  $\tilde{\Omega}$ ,  $\tilde{\Lambda} := \Lambda \subset \Omega \subset \tilde{\Omega}$ . We have :

- (i)  $\Lambda$  is dual bounded in  $A_{k-1}^p(\Omega)$  iff  $\tilde{\Lambda}$  is dual bounded in  $H^p(\tilde{\Omega})$ .
- (ii)  $\Lambda$  is  $A_{k-1}^p(\Omega)$  interpolating iff  $\tilde{\Lambda}$  is  $H^p(\tilde{\Omega})$  interpolating.
- (iii)  $\Lambda$  has the linear extension property in  $A_{k-1}^p(\Omega)$  iff  $\tilde{\Lambda}$  has the linear extension property in  $H^p(\tilde{\Omega})$ .

Proof.

For the (i) : suppose that  $\Lambda$  is dual bounded  $A_{k-1}^p(\Omega)$  and let  $\{\rho_a\}_{a \in \Lambda} \subset A_{k-1}^p(\Omega)$  be the dual sequence to the sequence  $\{b_{k-1,a}\}_{a \in \Lambda}$ ; extend it to  $\tilde{\Omega}$  :

$$\forall a \in \Lambda, \Gamma_a(z, w) := \rho_a(z),$$

then the subordination lemma gives us that  $\|\Gamma_a\|_{H^p(\tilde{\Omega})} \simeq \|\rho_a\|_{A_{k-1}^p(\Omega)}$  and we have, using corollary 2.3,

$$\forall a, c \in \Lambda, \langle \Gamma_a, k_{\tilde{c}} \rangle = \langle \Gamma_a, S((\cdot, 0), \tilde{c}) \rangle = \langle \rho_a, B(\cdot, c) \rangle = \langle \rho_a, b_{k-1,c} \rangle = \delta_{a,c} \|b_{k-1,c}\|_{p'},$$

Because  $\Lambda$  is dual bounded in  $A_{k-1}^p(\Omega)$ . Then we have, by corollary 2.3,

$$(4.2) \quad \forall \tilde{c} = (c, 0), \quad c \in \Omega, \quad \|b_{k-1,c}\|_{A_{k-1}^p(\Omega)} \simeq \|k_{\tilde{c}}\|_{H^p(\Omega)},$$

hence

$$\forall a, c \in \Lambda, \quad \langle \Gamma_a, k_{\tilde{c}} \rangle = \delta_{ac} \|b_{k-1,c}\|_{p'} \simeq \delta_{ac} \|k_{\tilde{c}}\|_{p'}$$

hence  $\tilde{\Lambda}$  is dual bounded in  $H^p(\tilde{\Omega})$ .

Because we used only equivalences in this proof, it works also for the converse, hence if  $\tilde{\Lambda}$  is dual bounded in  $H^p(\tilde{\Omega})$  then  $\Lambda$  is dual bounded in  $A_{k-1}^p(\Omega)$ .

For the (ii) : suppose that  $\tilde{\Lambda}$  is interpolating in  $H^p(\tilde{\Omega})$ . We want to show that  $\Lambda$  is  $A_{k-1}^p(\Omega)$  interpolating, so let  $\mu = \{\mu_a\}_{a \in \Lambda} \in \ell^p(\Lambda)$  the sequence to be interpolated. Set

$$\lambda = \{\lambda_{\tilde{a}}\}_{\tilde{a} \in \tilde{\Lambda}} \text{ with } \forall \tilde{a} \in \tilde{\Lambda}, \quad \lambda_{\tilde{a}} := \mu_a \times \frac{\|b_{k-1,a}\|_{A_{k-1}^{p'}(\Omega)}}{\|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})}};$$

then  $\lambda \in \ell^p(\tilde{\Lambda})$ ,  $\|\lambda\|_p \simeq \|\mu\|_p$  by (4.2).

Let  $F \in H^p(\tilde{\Omega})$  be the function making the interpolation of the sequence  $\lambda$ , which exists because  $\tilde{\Lambda}$  is  $H^p(\tilde{\Omega})$  interpolating. It means that

$$F(\tilde{a}) = \lambda_{\tilde{a}} \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})} = \mu_a \|b_{k-1,a}\|_{A_{k-1}^{p'}(\Omega)}.$$

Set  $\forall z \in \Omega$ ,  $f(z) := F(z, 0)$  then we have

$$\forall a \in \Lambda, \quad f(a) = F(a, 0) = F(\tilde{a}) = \mu_a \|b_{k-1,a}\|_{A_{k-1}^{p'}(\Omega)}.$$

Hence  $\Lambda$  is  $A_{k-1}^p(\Omega)$  interpolating.

Again the converse is straightforward because we use only equivalences.

For the (iii) : suppose that  $\tilde{\Lambda}$  has the bounded extension linear property, i.e. there is a linear operator  $\tilde{E} : \ell^p(\tilde{\Lambda}) \rightarrow H^p(\tilde{\Omega})$  such that  $F(z, w) := \tilde{E}(\lambda)(z, w)$ ,

$$F \in H^p(\tilde{\Omega}), \quad \forall a \in \Lambda, \quad F(\tilde{a}) = \mu_a \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})}, \quad \|F\|_{H^p(\tilde{\Omega})} \lesssim \|\mu\|_p.$$

With the same notations  $\lambda$  and  $\mu$  as above, set  $f(z) := F(z, 0) = \tilde{E}(\mu)(z, 0) =: E(\lambda)(z)$ , then clearly  $E$  is linear in  $\lambda$  and then still using the subordination lemma we have

$$\|f\|_{A_{k-1}^p(\Omega)} \lesssim \|F\|_{H^p(\tilde{\Omega})} \lesssim \|\mu\|_p \simeq \|\lambda\|_p$$

and

$$\forall a \in \Lambda, \quad f(a) = \mu_a \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})} = \lambda_{\tilde{a}} \|b_{k-1,a}\|_{A_{k-1}^{p'}(\Omega)}.$$

Hence  $\lambda \rightarrow E(\lambda)$  is bounded from  $\ell^p(\Lambda)$  in  $A_{k-1}^p(\Omega)$  and  $\Lambda$  is  $A_{k-1}^p(\Omega)$  interpolating with the linear extension.

Again the converse is straightforward. ■

#### 4.3. Application to strictly pseudo-convex domains.

In [5] we proved a general theorem on interpolating sequences in the spectrum of a uniform algebra. In the case of strictly pseudo-convex domains, it says that :

**Theorem 4.6.** *If  $\Omega$  is a strictly pseudo-convex domain in  $\mathbb{C}^n$  and if  $\Lambda \subset \Omega$  is a dual bounded sequence of points in  $H^p(\Omega)$ , then, for any  $q < p$ ,  $\Lambda$  is  $H^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

We have, as a consequence of the subordination lemma the following theorem.

**Theorem 4.7.** *Let  $\Omega$  be a strictly pseudo-convex domain in  $\mathbb{C}^n$  and  $\Lambda \subset \Omega$  be a dual bounded sequence of points in  $A_k^p(\Omega)$ , then, for any  $q < p$ ,  $\Lambda$  is  $A_k^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

Proof.

Let  $\tilde{\Omega}$  be the lift of  $\Omega$  in  $\mathbb{C}^{n+k+1}$  and  $\tilde{\Lambda} \subset \tilde{\Omega}$  the sequence  $\Lambda$  viewed in  $\tilde{\Omega}$ . We apply theorem 4.5 (i) to have that  $\tilde{\Lambda}$  is dual bounded in  $H^p(\tilde{\Omega})$  because  $\Lambda$  is dual bounded in  $A_k^p(\Omega)$ . Now we apply theorem 4.6 to get that  $\tilde{\Lambda}$  is  $H^q(\tilde{\Omega})$  interpolating with  $q < p$ , and has the bounded linear extension property, provided that  $p = \infty$  or  $p \leq 2$ . Then again theorem 4.5 (iii) to get the same for  $\Lambda$  in  $A_k^p(\Omega)$ . ■

We have a better result for the unit ball in  $\mathbb{C}^n$  : in [6] we proved

**Theorem 4.8.** *If  $\Lambda$  is a dual bounded sequence in the unit ball  $\mathbb{B}$  of  $\mathbb{C}^n$  for the Hardy space  $H^p(\mathbb{B})$ , then for any  $q < p$ ,  $\Lambda$  is  $H^q(\mathbb{B})$  interpolating with the bounded linear extension property.*

So copying the proof of theorem 4.7, just replacing theorem 4.6 by theorem 4.8 we get :

**Theorem 4.9.** *Let  $\Lambda$  be a dual bounded sequence in the unit ball  $\mathbb{B}$  of  $\mathbb{C}^n$  for the Bergman space  $A_k^p(\mathbb{B})$ , then for any  $q < p$ ,  $\Lambda$  is  $A_k^q(\mathbb{B})$  interpolating with the bounded linear extension property.*

**Remark 4.10.** *If we apply this theorem in the unit disc  $\mathbb{D}$  of  $\mathbb{C}$  we get that if  $\Lambda$  is a dual bounded sequence in  $A_k^p(\mathbb{D})$  then it is interpolating in  $A_k^q(\mathbb{D})$  for any  $q < p$ . In this particular case, one variable, the Schuster-Seip theorem [21] says that we have the interpolation up to  $q = p$ .*

#### 4.4. Application to convex domains of finite type.

To apply the general theorem on interpolating sequences in the spectrum of a uniform algebra to the case of convex domains of finite type in  $\mathbb{C}^n$ , we need to have a precise knowledge of the good family of polydiscs associated to the domain and in [7], we proved

**Theorem 4.11.** *If  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$  and if  $\Lambda \subset \Omega$  is a dual bounded sequence of points in  $H^p(\Omega)$ , then, for any  $q < p$ ,  $\Lambda$  is  $H^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

Then, again, copying the proof of theorem 4.7, just replacing theorem 4.6 by theorem 4.11 we get :

**Theorem 4.12.** *If  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$  and if  $\Lambda \subset \Omega$  is a dual bounded sequence of points in  $A_{k-1}^p(\Omega)$  then, for any  $q < p$ ,  $\Lambda$  is  $A_{k-1}^q(\Omega)$  interpolating with the linear extension property, provided that  $p = \infty$  or  $p \leq 2$ .*

**Remark 4.13.** *We applied the subordination principle since 1978 [2], [3] essentially in this case. For instance in [2] we used it to show that the interpolating sequences for  $H^p(\mathbb{B})$ , with  $\mathbb{B}$  the unit ball in  $\mathbb{C}^n$ ,  $n \geq 2$ , are different for different values of  $p$ , opposite to the one variable case of  $H^p(\mathbb{D})$ .*

### 5. THE $H^p$ -CORONA THEOREM FOR BERGMAN SPACES.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ . We say that the  $H^p$  -Corona theorem is true for  $\Omega$  if we have :

$$\forall g_1, \dots, g_k \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then

$$\forall f \in H^p(\Omega), \exists (f_1, \dots, f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

In the same vein, we say that the  $A_{k-1}^p(\Omega)$  -Corona theorem is true for  $\Omega$  if we have :

$$(5.3) \quad \forall g_1, \dots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m |g_j(z)| \geq \delta > 0$$

then



$$\forall f \in A_{k-1}^p(\Omega), \exists (f_1, \dots, f_m) \in (A_{k-1}^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j.$$

We then have

**Theorem 5.1.** *Suppose that the  $H^p$  -Corona is true for the domain  $\tilde{\Omega}$ , then the  $A_{k-1}^p(\Omega)$  -Corona theorem is also true for  $\Omega$ .*

Proof.

Let  $\tilde{\Omega}$  be the lifted domain ; then set

$$\forall j = 1, \dots, m, g_j \in H^\infty(\Omega), f \in H^p(\Omega), G_j(z, w) := g_j(z), F(z, w) := f(z).$$

Clearly the  $G_j$  are in  $H^\infty(\tilde{\Omega})$  and by the subordination lemma,  $F \in H^p(\tilde{\Omega})$ . Moreover, if the condition (5.3) is true, we have  $\forall (z, w) \in \tilde{\Omega}, \sum_{j=1}^m |G_j(z, w)| \geq \delta$  with the same  $\delta$ . So we can apply the hypothesis :

$$\exists (F_1, \dots, F_m) \in (H^p(\tilde{\Omega}))^m :: F = \sum_{j=1}^m F_j G_j.$$

Now set  $f_j(z) = F_j(z, 0)$  then applying again the subordination lemma, we have

$$f(z) = F(z, 0) = \sum_{j=1}^m F_j(z, 0) G_j(z, 0) = \sum_{j=1}^m f_j(z) g_j(z). \quad \blacksquare$$

### 5.1. Application to pseudo-convex domains.

**Corollary 5.2.** *We have the  $A_{k-1}^p(\Omega)$  -Corona theorem in the following cases :*

- with  $p = 2$  if  $\Omega$  is a bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;
- with  $1 < p < \infty$  if  $\Omega$  is a bounded strictly pseudo-convex domain in  $\mathbb{C}^n$ .

The first case because Andersson [10] (with a preprint in 1990) proved the  $H^2$  Corona theorem for  $\Omega$  bounded weakly pseudo-convex domain in  $\mathbb{C}^n$ ;

the last one for two generators because we proved [4] ( with [9] already in 1980) the  $H^p$  Corona theorem for two generators in the ball ; for any number of generators because Andersson & Carlsson [11] (see also [8]) proved the  $H^p$  Corona theorem in this case. \blacksquare

## 6. ZEROS SET OF THE NEVANLINNA-BERGMAN CLASS.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $u$  a holomorphic function in  $\Omega$ . Set  $X := \{z \in \Omega :: u(z) = 0\}$  the zero set of  $u$  and  $\Theta_X := \partial \bar{\partial} \log |u|$  its associated  $(1, 1)$  current of integration.

**Definition 6.1.** *An analytic set  $X := u^{-1}(0)$ ,  $u \in \mathcal{H}(\Omega)$ , in the domain  $\Omega$  is in the Blaschke class,  $X \in \mathcal{B}(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \left| \int_{\Omega} (-r(z)) \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty,$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}(\Omega)$  then it is well known [24] that  $X$  is in the Blaschke class of  $\Omega$ .

We do the analogue for the Bergman spaces :

**Definition 6.2.** *An analytic set  $X := u^{-1}(0)$ ,  $u \in \mathcal{H}(\Omega)$ , in the domain  $\Omega$  is in the Bergman-Blaschke class,  $X \in \mathcal{B}_{k-1}(\Omega)$ , if there is a constant  $C > 0$  such that*

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty,$$

where  $\Lambda_{n-1, n-1}^\infty(\bar{\Omega})$  is the space of  $(n-1, n-1)$  continuous form in  $\bar{\Omega}$ , equipped with the sup norm of the coefficients.

If  $u \in \mathcal{N}_{k-1}(\Omega)$  then  $X$  is in the Bergman-Blaschke class of  $\Omega$ , for instance again by use the subordination lemma from the case  $\mathcal{N}(\tilde{\Omega})$ .

Hence exactly as for the Corona theorem we can set the definitions :

we say that the *Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

And the same for the Bergman spaces :

we say that the *Bergman-Blaschke characterization is true for  $\Omega$*  if we have :

$$X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega) \text{ such that } X = \{z \in \Omega :: u(z) = 0\}.$$

**Theorem 6.3.** *Suppose that the Blaschke characterization is true for the lifted domain  $\tilde{\Omega}$ , then the Bergman-Blaschke characterization is also true for  $\Omega$ .*

Proof.

Let  $\tilde{\Omega}$  be the lifted domain in  $\mathbb{C}^{n+k}$  of  $\Omega$  ; then set  $X = u^{-1}(0)$ ,  $\Theta_X$  its associated current and suppose that  $X \in \mathcal{B}_k(\Omega)$ .

This means that

$$\forall \beta \in \Lambda_{n-1, n-1}^\infty(\bar{\Omega}), \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \beta \right| \leq C \|\beta\|_\infty.$$

Let

$$\forall w \in \mathbb{C}^k, U(z, w) := u(z), \tilde{X} := U^{-1}(0) \cap \tilde{\Omega} \subset \tilde{\Omega}, \tilde{\Theta}_{\tilde{X}} = \partial \bar{\partial} \log |U| ;$$

we shall show that  $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$ . We have that  $\tilde{\Theta}_{\tilde{X}}$  does not depend on  $w$ , hence

$$\forall \tilde{\beta} \in \Lambda_{n+k-1, n+k-1}^\infty(\bar{\tilde{\Omega}}), A := \int_{\tilde{\Omega}} (-\tilde{r}(z, w)) \tilde{\Theta}_{\tilde{X}} \wedge \tilde{\beta} = \int_{\Omega} \Theta_X(z) \wedge \int_{|w|^2 < -r(z)} -(r(z) + |w|^2) \tilde{\beta}(z, w).$$

Because  $\Theta_X$  is a  $(1, 1)$  current depending only on  $z$ , this means that in the integral in  $w$  we have only the terms containing  $dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_k \wedge d\bar{w}_k$ , the other terms being 0 against  $\Theta_X$ . So this integral in  $w$  gives a  $(n-1, n-1)$  form in  $z$ .

Now set

$$\beta_1(z) := \int_{|w|^2 < -r(z)} \left(1 + \frac{|w|^2}{-r(z)}\right) \tilde{\beta}(z, w),$$

we have

$$A = \int_{\Omega} \Theta_X(z) \wedge (-r(z)) \beta_1(z)$$

and, because  $1 + \frac{|w|^2}{-r(z)} < 2$  in  $\{|w|^2 < -r(z)\}$ , we have

$$|\beta_1(z)| \leq 2 \|\tilde{\beta}\|_\infty \int_{|w|^2 < -r(z)} dm_k(w) \leq 2v_k \|\tilde{\beta}\|_\infty (-r(z))^k,$$

because we get the volume in  $\mathbb{C}^k$  of the ball centered in 0 and of radius  $\sqrt{-r(z)}$ .

Set  $\beta_2(z) := (-r(z))^{-k} \beta_1(z)$ , we have  $\|\beta_2\|_\infty \leq 2v_k \|\tilde{\beta}\|_\infty$  and

$$A = \int_{\Omega} \Theta_X(z) \wedge (-r(z)) \beta_1(z) = \int_{\Omega} \Theta_X(z) \wedge (-r(z))^{k+1} \beta_2(z).$$

We can apply the hypothesis  $X \in \mathcal{B}_{k-1}(\Omega)$  to the integral  $A$  :

$$|A| \leq \|\beta_2\|_\infty \lesssim 2\|\tilde{\beta}\|_\infty,$$

hence  $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$ .

Now we apply the hypothesis of the theorem,

$$\exists V \in \mathcal{N}(\tilde{\Omega}) :: \tilde{X} = V^{-1}(0),$$

and clearly  $X = V^{-1}(0) \cap \{w = 0\}$ , because if  $z \in X$  then  $\forall w :: |w|^2 < -r(z)$ ,  $(z, w) \in \tilde{X}$ . Hence we set

$$v(z) := V(z, 0) \in \mathcal{N}_{k-1}(\Omega),$$

by the subordination lemma, and we are done. ■

### 6.1. Application to pseudo-convex domains.

**Corollary 6.4.** *The Bergman-Blaschke characterization is true in the following cases :*

- if  $\Omega$  is a strictly pseudo-convex domain in  $\mathbb{C}^n$  ;
- if  $\Omega$  is a convex domain of finite type in  $\mathbb{C}^n$ .

Proof.

The first case is true by the famous theorem proved by Henkin [19] and Skoda [24] which says that the Blaschke characterization is true for strictly bounded pseudo-convex domain in  $\mathbb{C}^n$ .

The second one because the Blaschke characterization is true for convex domain of finite strict type by a theorem of Bruna-Charpentier-Dupain [12] generalized to all convex domains of finite type by Cumenge [17] and Diederich & Mazzilli [18]. ■

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